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Exercise Sheet 6 for
Algorithm Engineering, SS 14

Hand In: *Until Monday, 02.06.2014, 10:00 am, email to* `wild@cs...` *or in lecture.*

Problem 13

3 + 3 points

Use *Darboux's theorem* (as given in class) to compute leading term asymptotics for the coefficients of the following generating functions.

Precisely argue why Darboux's theorem is applicable and how you compute the asymptotics. As always, sensible use of computer algebra is allowed — if not recommended — but make sure to comment on intermediate steps of your computation.

a) $A(z) = \sum_{n\ge 0} a_n z^n = \dfrac{1 - \sqrt[3]{-3z^2 - 2z + 1}}{1 + z}$.

b) $B(z) = \sum_{n\ge 0} b_n z^n = \dfrac{1 - \sqrt{1 - 2z}}{4z^2 - 5z + 1}$.

Problem 14

3 points

Give an asymptotic for $[z^n]A(z)$ using $\mathcal{O}$-transfer for

$$A(z) = \frac{1 - \sqrt{1 - 7z}}{2z}.$$

Your asymptotics should be as precise as possible using the table from the lecture notes!

Problem 15

4 + 1 + 2 + 1 + 2 points

For $n \in \mathbb{N}$ and $r \in \mathbb{R}$ let $\mathcal{H}_n^{(r)}$ be the *generalized harmonic numbers* defined by

$$\mathcal{H}_n^{(r)} \;:=\; \sum_{i=1}^{n} \frac{1}{i^r} \,.$$

The ordinary harmonic numbers thus correspond to $r = 1$: $\mathcal{H}_n = \mathcal{H}_n^{(1)}$.

a) Prove the following asymptotic for the generalized harmonic numbers

$$\mathcal{H}_n^{(r)} \;\sim\; \begin{cases} \log n, & \text{if } r = 1; \\ c_r, & \text{if } r > 1; \\ \dfrac{n^{1-r}}{1-r}, & \text{otherwise.} \end{cases} \qquad (\mathcal{H})$$

Here, $c_r \in \mathbb{R}_{>1}$ is some real constant depending on r.
(Can you express c_r in terms of a well-known function?)

Hint: Use integrals to obtain asymptotically matching lower and upper bounds on $\mathcal{H}_n^{(r)}$. This technique works for both cases.

b) In class, we introduced the 80-20 rule for discrete probability distributions with decreasing weights $p_1 \ge p_2 \ge \cdots$ as

$$w(k) \;:=\; \frac{p_1 + \cdots + p_k}{p_1 + \cdots + p_{5k}} \;=\; 0.8 \qquad \text{for all } k\,, \qquad \text{(80-20)}$$

i. e., among the $5k$ most important items, 80 % of the weight is contributed by the k most important items (among them).

We also gave a power law distribution that "approximately" fulfils the 80-20 rule:

$$p_i := \frac{c}{i^{1-\theta}} \qquad \text{for } 1 \le i \le n \qquad \text{with } c = 1/\mathcal{H}_n^{(1-\theta)} \text{ and } \theta = \log_{0.2}(0.8) \approx 0.1386$$

Compute the ratio $w(k)$ for $n = 10$ and $k = 2$ and for $n = 100$ and $k = 20$.
Is it close to 0.8?

c) Compute a general closed expression for $w(k)$ for the probability distribution from b) and use it to show that

$$\lim_{k\to\infty} \frac{p_1 + \cdots + p_k}{p_1 + \cdots + p_{5k}} \;=\; 0.8 \qquad \text{(assuming } n \ge 5k) \,.$$

Hint: Use $(\mathcal{H})$.

d) Assume the alternative probability distribution for the keys $1, \ldots, n$ given by

$$p'_i \;:=\; \frac{i^\theta - (i-1)^\theta}{n^\theta} \qquad\qquad \text{for } 1 \le i \le n,$$

(where $\theta = \log_{0.2}(0.8)$ as before).
Show that this distribution fulfils the 80-20 rule *exactly*, i. e., for all $k \le \lfloor n/5 \rfloor$.

e) Determine the expected search costs for a (successful) search in a list of length n, where the ith element is requested with probability p'_i.